

Nonlinear Landau Damping of Alfvén Waves in a High β Plasma *

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To Professor Arnulf Schlüter on his 60th Birthday

A study is made of the nonlinear damping of parallel propagating Alfvén waves in a high β plasma. Two circularly polarized parallel propagating waves give rise to a beat wave, which in general contains both a longitudinal electric field component and a longitudinal gradient in the magnetic field strength. The wave damping is due to the interactions of thermal particles with these fields. If the amplitudes of the waves are low, a given wave (ω_1, k_1) is damped by the presence of all longer wavelength waves; thus, if the amplitudes of the waves in the wave spectrum increase with wave length, the effect of the longest waves is dominant.

However, when the amplitude of the waves is sufficiently high, the particles are trapped in the wave packets, and the damping rate may be considerably reduced. We calculate the induced electrostatic field, and examine the trapping of thermal particles in a pair of waves. Finally, we give examples of modified damping rates of a wave in the presence of a spectrum of waves, and show that, when the trapping is effective, the waves are mostly damped by their interactions with waves of comparable wavelengths.

1. Introduction

The damping of hydromagnetic waves in fully ionized, collisionless plasmas is an important process, relevant to many astrophysical situations, where these waves may serve to heat the gas, to transport energy over large distances, or to scatter energetic particles. In our immediate environment, the solar wind is one example of such a plasma. Another interesting case is the hot component of the interstellar medium, which has a temperature of $\sim 10^6$ K and a density of $\sim 10^{-3} \text{ cm}^{-3}$, and is believed to occupy a large fraction of the galactic volume. The strength of the magnetic field in this medium is not known; but it seems most likely that the ratio β of plasma pressure to magnetic pressure exceeds unity. It is presumably in this medium that galactic cosmic rays are trapped by resonant scattering off hydromagnetic waves, and possibly also accelerated by diffusive compression in large scale shock waves (see Cesarsky [1] for a review).

Both in the solar wind, where the waves are produced by an external source, and in the inter-

stellar medium, where they may largely be due to cosmic ray pressure gradients, the waves predominantly propagate parallel to each other. Thus, in this paper, we concentrate on parallel wave fields in high- β plasmas.

Compressive hydromagnetic waves are damped by transit time damping on the thermal particles (Stepanov [2]; Barnes [3]). This occurs through the Landau resonance $\omega - k_{\parallel} \cdot v_{\parallel} = 0$, where ω , k_{\parallel} , and v_{\parallel} denote the wave frequency, and the components parallel to the average magnetic field of the wave vector, and particle velocity, respectively. Since for $\beta > 1$ the number of resonant particles becomes large, strong damping results in this case. Alfvén waves propagating obliquely to the field are coupled to the compressive modes for high β , and are therefore also damped (Foote and Kulsrud [4]). But parallel propagating Alfvén waves are purely transverse. There is only the cyclotron resonance $\omega - k_{\parallel} v_{\parallel} - \Omega = 0$, where Ω is the particle gyrofrequency, which cannot be fulfilled by a thermal particle, and there is no linear damping of any importance. To higher order in the wave amplitudes, however, there can be an effect: two circularly polarized parallel propagating waves give rise to a beat wave which in general contains both a

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longitudinal field component and a longitudinal gradient in the magnetic field strength. This results in nonlinear Landau damping, both electrostatic and magnetostatic i.e. transit time damping. Thus, for example, one can possibly have a steady state, with wave excitation by cyclotron resonance of anisotropic energetic particles, i.e. cosmic rays (Lerche [5]; Wentzel [6]; Kulsrud and Pearce [7]), and nonlinear damping by the thermal plasma.

That there should be an electrostatic field in second order for a linearly polarized Alfvén wave was pointed out by Hollweg [8] who then proceeded to calculate the electrostatic Landau damping of this wave. Due to his fluid approximation for the wave field he obtained spurious resonances. The general damping for two arbitrarily polarized circular Alfvén waves was calculated by Lee and Völk [9]. Later Kulsrud [10] considered the nonlinear transit time damping for a linearly polarized wave only; he pointed out that the damping should be modified by particle trapping in the second order magnetic mirrors which have a finite coherence time at finite frequencies. Cesarsky and Kulsrud [11] took this trapping effect into account to calculate the cosmic ray self confinement in a high β hot interstellar medium. The unsaturated damping rate was applied by Blandford [12], Völk et al. [13], Völk and McKenzie [14] and McKenzie [15] to the question of wave amplitudes in diffusive shock acceleration of cosmic rays.

In this paper we shall consider nonlinear Landau damping in the presence of many waves and calculate the trapping in the combined electrostatic and magnetic mirror fields of the beat waves. This results in a modified growth rate taking into account the finite wave dispersion.

2. Unmodified Damping Rate in the Many Wave Case

The general case of nonlinear Landau damping of an Alfvén wave in the presence of another Alfvén wave has been calculated by Lee and Völk [9]. We are interested here in the case where $\beta \gtrsim 1$ and where the waves propagate parallel to each other, excited by the cosmic ray streaming instability.

Assuming the thermal gas to be Maxwellian in the velocity component parallel to \mathbf{B}_0 , with temperature T_{\parallel} , and isotropic if not necessarily Maxwellian regarding the perpendicular velocity, the

damping rate for parallel waves, when

$$\beta_{\parallel p} = 4\pi n K T_{\parallel} / B_0^2 \gtrsim 10^{-1},$$

is given by:

$$\Gamma_1 = - \frac{\omega_1 \omega_0}{4 |\omega_0|} \frac{B_2^2}{B_0^2} (2\pi\beta_{\parallel p})^{1/2} \cdot [(\xi_p^2/2) \{L_p + \frac{1}{4}(1 + \xi_e/\xi_p)^2\}], \quad (1)$$

where Γ_1 is the damping rate of wave 1 with frequency ω_1 ; to first approximation $\omega = k V_A$ where V_A is the Alfvén velocity. The field amplitude of wave 2 is given by B_2 , and its frequency by ω_2 . We have $\omega_0 = \omega_1 - \omega_2$. The indices p and e refer to protons and electrons, respectively; we consider for simplicity here a pure hydrogen plasma. In addition

$$\xi_p = \langle V_{\perp}^2 \rangle / (2k T_{\parallel} / m_p), \quad (2)$$

$$L_p = \langle V_{\perp}^4 \rangle / (2 \langle V_{\perp}^2 \rangle^2) \quad (3)$$

with analogous expressions for electrons.

Right hand (left hand) polarization is given by $\omega_1 > 0$ ($\omega_1 < 0$), and analogous prescriptions hold for wave 2. It is worth noting that the term $(1/4) \cdot (1 + \xi_e/\xi_p)^2$ in the brace of (1) corresponds to electrostatic Landau damping in the driven electrostatic ion sound beat mode. For a thermal plasma it equals the term L_p corresponding to transit time damping (Barnes [3]) in the magnetic field strength variation of the beat wave. Thus, for a thermal plasma the two types of damping contribute equally to Γ_1 . Kulsrud [11] has given a relation similar to (1), but containing only the first term in the curly bracket on the r.h.s. corresponding to the transit time damping effect.

For a thermal plasma, the square bracket at the r.h.s. of (1) equals unity. Defining the proton velocity $V_p = V_A (\beta_p)^{1/2}$, we simply have:

$$\Gamma_1 = - \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{\omega_0}{|\omega_0|} |k_1| V_p \frac{B_2^2}{B_0^2}. \quad (4)$$

We see from (1) and (4) that $\Gamma_1 < 0$ if $\text{sign}(\omega_1) \neq \text{sign}(\omega_2)$, or if both waves have the same polarization but $|\omega_1| > |\omega_2|$. A wave of frequency ω_1 grows if it interacts with a wave 2 of the same polarization, and with $|\omega_2| > |\omega_1|$. Therefore, $\Gamma_1 / (k_1 |B_2|^2)$ depends only on the polarizations of the two waves, but not on the values of k_1 and k_2 .

Let us now consider the effect, on waves of wave-number k_1 , of a whole spectrum of waves. We will

assume throughout that, at each wavenumber, the spectrum contains an equal number of waves of each polarization. To generalize (1), we replace B_2^2 by $P(k_2)$, and integrate over the spectrum. $P(k_2)/8\pi$ represents the wave power spectrum per logarithmic bandwidth. The damping decrement of a wave with wavenumber k_1 is:

$$\begin{aligned} \Gamma(k_1)/|k_1| &= -\frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{V_p}{B_0^2} \left[\frac{1}{2} \int_{k_0}^{\infty} P(k') d(\log k') \right. \\ &\quad \left. + \frac{1}{2} \int_{k_0}^{k_1} P(k') d(\log k') - \frac{1}{2} \int_{k_1}^{\infty} P(k') d(\log k') \right] g \\ &= -\frac{\sqrt{\pi}}{4\sqrt{2}} \cdot \frac{V_p}{B_0^2} g \int_{k_0}^{k_1} P(k') d(\log k'). \end{aligned} \quad (5)$$

k_0 corresponds to the longest excited wavelength of the spectrum and g represents the expression in brackets in (1). From this simple expression it obviously follows that a given wave ω_1 , k_1 is damped by the presence of all longer wavelength waves. Thus short wavelength waves are damped most strongly.

There is a problem here in that all the longer wavelengths come in formally, although from some large scale on, all perturbations should be rather counted as (average) background fields B_0 . We will see in the following that the "infrared catastrophe" of formula (5) disappears when we take into account the saturation of the damping mechanism when the wave power is high.

3. Calculation of the Induced Electrostatic Field

As argued on the basis of double adiabatic fluid theory by Hollweg [8] and calculated with the use of the Vlasov equation, two parallel Alfvén waves create in second order an electrostatic field perturbation E_{\parallel} which leads to nonlinear electrostatic Landau damping on the thermal particles. This field E_{\parallel} is calculated below.

According to Lee and Völk [9], the Fourier mode $E_{\parallel, k} = \int dz dt E_{\parallel}(z, t) \exp\{-i(kz - \omega t)\}$ (6)

of the induced field component $E(z, t)$ parallel to the average magnetic field B_0 is given by:

$$E_{\parallel, k} = (D_{\parallel, k})^{-1} \int dk' d\omega' M_{k, k'} \cdot (B_{+, k'} \cdot B_{-, l} - \langle B_{+, k'} \cdot B_{-, l} \rangle) + O(E_{\pm}^4). \quad (7)$$

In (7) the quantities $B_{\pm, k} \pm iB_y, k$ denote the magnetic Fourier components of right (+) and left (−) handed circularly polarized Alfvén waves with (real) wave vector k and (real) frequency ω . The matrix

element $M_{k, k'}$ is given by

$$M_{k, k'} = \frac{i}{4kc} \sum_r \omega_r^2 \eta_r \int d^3v (\omega - kv_{\parallel})^{-1} \frac{\omega'(\omega - \omega')}{k'(k - k')} \cdot \left[H_{k'} \frac{(G_l f_r)}{(\omega - lv_{\parallel} + \Omega_r)} + H_l \frac{(G_{k'} f_r)}{(\omega' - kv_{\parallel} - \Omega_r)} \right], \quad (8)$$

where $l = k - k'$, $\omega = \omega - \omega'$; $\omega_r^2 = 4\pi q_r^2 n_r / m_r$, and $\eta_r = q_r / (m_r c (2\pi)^2)$; the particle mass, charge, and gyrofrequency as well as the number density, and the speed of light, are denoted by q_r , m_r , $\Omega_r = q_r B_0 / (m_r c)$, n_r , and c , respectively*. We define the operators

$$G_k = (1 - kv_{\parallel}/\omega) \partial / \partial v_{\perp} + (kv_{\perp}/\omega) \partial / \partial v_{\parallel},$$

$$A_k = (1 - kv_{\parallel}/\omega) v_{\perp}, \quad \text{and} \quad H_k = G_k + A_k.$$

The longitudinal dielectric constant $D_{\parallel, k}$ is given by

$$D_{\parallel, k} = 1 + \sum_r (\omega_r^2 / k) \int d^3v (\omega - kv_{\parallel})^{-1} \partial f_r / \partial v_{\parallel}. \quad (9)$$

The quantities $f_r(v_{\parallel}, v_{\perp})$ denote the distribution functions averaged over the ensemble of fluctuations with $\int d^3v f_r = 1$. We consider a spatially homogeneous ensemble, where the average is indicated by $\langle \rangle$.

Considering hydromagnetic waves with $kr_g \ll 1$, where r_g is the gyroradius of thermal particles, as well as $\beta_{\parallel p} \gg 1$, we neglect gyroresonance effects and expand the denominators of $M_{k, k'}$ in powers of $(\omega - kv_{\parallel})/\Omega_r \ll 1$, retaining only the lowest order terms.

If we now assume $B_{\pm, k}$ to represent two circularly polarized waves 1 and 2, we put in (7):

$$B_{-, k, \omega} = \tilde{B}_1 \delta(k - k_1) \delta(\omega - \omega_1) + \tilde{B}_2 \delta(k - k_2) \delta(\omega - \omega_2) \quad (10)$$

with $B_{+, k, \omega} = (B_{-, -k, -\omega})^*$, * denoting the complex conjugate, and $B_{1, 2}$ being the Fourier amplitudes of the modes 1 and 2. This result in:

$$\begin{aligned} B_{+, k'} \cdot B_{-, l} - \langle B_{+, k'} \cdot B_{-, l} \rangle &= \tilde{B}_1^* \tilde{B}_2 \delta(k + k_1 - k_2) \delta(k' + k_1) \\ &\cdot \delta(\omega + \omega_1 - \omega_2) \delta(\omega' + \omega_2) + (1 \leftrightarrow 2) \end{aligned} \quad (11)$$

where $(1 \rightarrow 2)$ denotes the interchange of the suffixes 1 and 2 in the preceding expression.

* The matrix element used in (8) is $(\omega/kc)^2$. Matrix element $M_{k, k'}$ of Lee and Völk (LV, [9]), since in (7) we have used $B_{\pm, k} = (kc/\omega) (\pm i) E_{\pm}$, instead of E_{\pm} as in LV (25). Also we have used $f_r = c^{-3} f_r^{(0)}$. Since everything is nonrelativistic, we also have $v_{\perp, \parallel} = cp_{\perp, \parallel}$, where p = dimensionless momentum of LV. Accordingly, our G_k , A_k , and H_k are c^{-1} times the corresponding quantities in LV.

Taking a Maxwellian distribution for the parallel velocities:

$$f_r = \left(\frac{m_r}{2\pi K T_{\parallel r}} \right)^{1/2} \exp \left\{ -\frac{m_r v_{\parallel}^2}{2 K T_{\parallel r}} \right\} f_{r\perp}(v_{\perp}) \quad (12)$$

with $\int_0^{\infty} 2\pi v_{\perp} dv_{\perp} f_{r\perp}(v_{\perp}) = 1$,

we obtain for the two lowest orders in $(\omega - kv_{\parallel})/\Omega_r$ for parallel propagating waves:

$$\begin{aligned} E_{\parallel, k} = & (-1) \frac{i(2\pi)^2}{2c B_0} \left[\frac{\omega_2}{k_2} - \frac{\omega_1}{k_1} \right] \\ & \cdot \left\{ \frac{\tilde{B}_1^* \tilde{B}_2}{(2\pi)^4} \delta(k - (k_2 - k_1)) \delta(\omega - (\omega_2 - \omega_1)) - \frac{\tilde{B}_1 \tilde{B}_2^*}{(2\pi)^4} \delta(k - (k_1 - k_2)) \delta(\omega - (\omega_1 - \omega_2)) \right\} \\ & + \left\{ \left(\frac{\omega}{k} \right)^2 \sum_r \frac{4\pi N q_r^2}{2 K T_{\parallel r}} + \left(\frac{\omega}{k} \right)^2 \frac{4\pi N q_p^2 \alpha_p}{2 K T_{\parallel p}} \left[-i\sqrt{\pi} \frac{k}{|k|} e^{-\alpha_p^2} + 2S(\alpha_p) \right] \right\}^{-1} \frac{ik}{c} \\ & \cdot \left\{ \sum_r \frac{\omega r^2 \eta_r \alpha_r \langle v_{\perp}^2 \rangle}{\Omega_r^2} (-1) + \frac{(2\pi)^4 \omega_p^2 \eta_p \langle v_{\perp}^2 \rangle \alpha_p^3}{\Omega_p^2} \left[-i\sqrt{\pi} \frac{k}{|k|} e^{-\alpha_p^2} + 2S(\alpha_p) \right] \right\} \\ & \cdot \frac{\tilde{B}_1^* \tilde{B}_2}{(2\pi)^4} \delta(k - (k_2 - k_1)) \delta(\omega - (\omega_2 - \omega_1)) + (1 \leftrightarrow 2). \end{aligned} \quad (13)$$

Here we have assumed $n_e = n_p = N$, and abbreviated $\alpha_r^2 = (2\beta_{\parallel, r})^{-1}$. In addition

$$S(x) = e^{-x^2} \int_0^x e^{t^2} dt,$$

and we have neglected all terms $O(\alpha_e)$ but kept all orders in α_p .

Although the first term in (13) is formally of zeroth order in ω/Ω_p , it is actually of first order, since $[\omega_2/k_2 - \omega_1/k_1]$ vanishes to zeroth order. Therefore, we must include Faraday rotation effects in the first order term:

$$\frac{\omega_1}{k_1} - \frac{\omega_2}{k_2} \cong \frac{2\beta_{\parallel p} V_A^2}{4\Omega_p} (k_1 - k_2), \quad (14)$$

where k is to be taken positive (negative) for a left (right) hand circularly polarized wave.

The expression (13) is rather cumbersome and we shall not use it in this form. We shall rather assume that $\langle v_{\perp}^2, p \rangle = 2 K T_{\parallel, r} / m_r$, and only keep the lowest non-vanishing term in α_p in the second term. Transforming back to z and t then yields:

$$\begin{aligned} E_{\parallel}(z, t) = & \frac{2\beta_{\parallel p} \tilde{B}_1 \tilde{B}_2}{16\pi N q_p (2\pi)^4} \frac{\partial}{\partial z} \\ & \cdot \{ \cos[(k_1 - k_2)z - (\omega_2 - \omega_1)t] \\ & + (-1) (\pi/2 \beta_{\parallel p})^{1/2} \text{sign}(k_2 - k_1) \\ & \cdot \sin[(k_2 - k_1)z - (\omega_2 - \omega_1)t] \\ & + O(\beta_{\parallel p}^{-1}) \}. \end{aligned} \quad (15)$$

The second term in the brace of (15) arises from the Landau damping of the beat mode. It leads to a phase shift of the induced electrostatic field analogous to a damped driven harmonic oscillator.

4. Trapping of Thermal Particles in a Pair of Waves

According to Lee and Völk [9], it is the protons who contribute primarily to the damping near $\beta_p \gg 1$. Let us then consider the protons ($q_p = e$).

The equation of motion for a proton interacting with the beat wave is:

$$m_p \frac{d^2 z}{dt^2} = e E_{\parallel} - \frac{m_p v_{\perp}^2}{2 B_0} \frac{\partial}{\partial z} |B|. \quad (16)$$

Taking for B a pair of circularly polarized waves $B^{(1,2)}$ propagating in a background field B_0 , we have:

$$\begin{aligned} |B| = & |B_0 + B^{(1)} + B^{(2)}| \\ = & |B_0| \left\{ 1 + \frac{2B_0(B^{(1)} + B^{(2)})}{B_0^2} + \frac{(B^{(1)})^2}{B_0^2} \right. \\ & \left. + \frac{(B^{(2)})^2}{B_0^2} + \frac{2(B^{(1)} B^{(2)})^{1/2}}{B_0^2} \right\} \end{aligned} \quad (17)$$

and

$$\frac{\partial}{\partial z} |B| = \frac{\partial}{\partial z} \frac{(B^{(1)} B^{(2)})}{B_0}.$$

For the case (10) and \tilde{B}_1, \tilde{B}_2 taken to be real, we obtain with the magnetic field amplitude

$$B_{1,2} \equiv \tilde{B}_{1,2}/(2\pi)^2:$$

$$\frac{\partial}{\partial z} |\mathbf{B}| = - \frac{B_1 B_2}{|\mathbf{B}_0|} \frac{\partial}{\partial z} \cdot \cos \{(k_2 - k_1)z - (\omega_2 - \omega_1)t\} \quad (18)$$

which results in

$$m_p \frac{d^2 z}{dt^2} = B_1 B_2 \left[\frac{2\beta_{\parallel p}}{16\pi N} \frac{\partial}{\partial z} \cdot \left\{ \cos [(k_2 - k_1)z - (\omega_2 - \omega_1)t] + (-1) \left(\frac{\pi}{2\beta_{\parallel p}} \right)^{1/2} \cdot \sin [(k_2 - k_1)z - (\omega_2 - \omega_1)t] \right\} + \frac{m_p v_{\perp}^2}{2B_0^2} \frac{\partial}{\partial z} \cos [(k_2 - k_1)z - (\omega_2 - \omega_1)t] \right] \quad (19)$$

or, in an abbreviated, self-evident notation:

$$\frac{d^2 z}{dt^2} = B_1 B_2 \frac{2\beta_{\parallel p}}{16\pi N m_p} \left[\frac{\partial}{\partial z} \left\{ \cos - \left(\frac{\pi}{2\beta_{\parallel p}} \right)^{1/2} \sin \right\} + \frac{m_p v_{\perp}^2}{K T_{\parallel}} \frac{\partial}{\partial z} \cos \right]. \quad (20)$$

Disregarding the phase shift due to the damping term $\propto \sin$, we obtain for the characteristic frequency of the trapped particle motion ($t_{\text{trap}} \equiv \text{trap-}$ time):

$$\frac{2\pi}{t_{\text{trap}}} = \left\{ \frac{B_1 B_2}{B_0^2} \frac{K T_{\parallel}}{2 m_p} |k_2 - k_1|^2 \right\}^{1/2} \cdot \left[1 + \frac{m_p v_{\perp}^2}{K T_{\parallel}} \right]^{1/2}. \quad (21)$$

Since $a \sin \varphi + b \cos \varphi = (a^2 + b^2)^{1/2} \cos(\varphi - \psi)$, where $\psi = \arctg(a/b)$ is the resulting phase shift, the inclusion of the sin-term in (20) leads to the actual trapping time:

$$\frac{2\pi}{t_{\text{trap}}} = \left\{ \frac{B_1 B_2}{B_0^2} \frac{K T_{\parallel}}{2 m_p} |k_2 - k_1|^2 \right\}^{1/2} \cdot \left[\left\{ 1 + \frac{m_p v_{\perp}^2}{K T_{\parallel}} \right\}^2 + \frac{\pi}{2\beta_{\parallel p}} \right]^{1/4}. \quad (22)$$

Clearly, the term $\propto v_{\perp}^2$ in the square bracket of the r.h.s. of (22) is due to the magnetostatic trapping, whereas the two other terms are due to electro-

static trapping. Thus low- v_{\perp} particles are trapped electrostatically whereas high- v_{\perp} particles are trapped magnetically.

The case $k_2 = -k_1$, $B_2 = B_1$, corresponding to a linearly polarized wave, gives

$$\frac{2\pi}{t_{\text{trap}}} = \left\{ \frac{B_1^2}{B_0^2} \frac{K T_{\parallel}}{m_p} 2 k_1^2 \right\}^{1/2} \cdot \left[\left\{ 1 + \frac{m_p v_{\perp}^2}{K T_{\parallel}} \right\}^2 + \frac{\pi}{2\beta_{\parallel p}} \right]^{1/4} \quad (23)$$

which suitably generalizes Kulsrud's [10] result.

5. Saturated Nonlinear Damping Rate

Once the particles are trapped in the wave-packet, they no longer exchange energy with it on a secular basis. Thus, when the correlation time t_c of the two waves is longer than the trapping time t_{trap} , the damping rate of (1) is roughly multiplied by a factor t_{trap}/t_c .

For two waves (ω_1, k_1) and (ω_2, k_2) , the correlation time is:

$$t_c = 2\pi \left| \left(\frac{1}{k_1} - \frac{1}{k_2} \right) \right| \left| \left(\frac{\omega_1}{k_1} - \frac{\omega_2}{k_1} \right) \right| = \left\{ \frac{|k_1 k_2|}{2\pi} \frac{V_p^2}{2\Omega_p} \right\}^{-1}. \quad (24)$$

The saturation condition can be written:

$$\frac{|k_2 - k_1|}{|k_2|} > \frac{B_0}{\sqrt{B_1 B_2}} \frac{V_p}{\Omega_p} \frac{|k_1|}{h}, \quad (25)$$

where $h^4/4$ represents the expression in brackets in (22) and (23). When inequality (25) is fulfilled, the damping rate resulting from the interaction with another wave becomes:

$$\Gamma_s(k_1) = (-1) \cdot \frac{\omega_0}{|\omega_0|} \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{V_p^2}{\Omega_p} \frac{B_2^{3/2}}{B_0 B_1^{1/2}} \frac{g}{h} \frac{k_1^2 |k_2|}{|k_2 - k_1|}. \quad (26)$$

Let us now consider the damping rate of waves (ω_1, k_1) in the presence of a wave spectrum. Clearly, in general, the appropriate damping rate is given by (1) for some waves in the spectrum, and by (26) for others. There is no need to give here cumbersome formulae that would encompass every case. Let us

rather concentrate on the interesting limit:

$$\frac{k_1 V_p}{\Omega_p} \frac{B_0}{B_1} \equiv \varepsilon \ll 1 \quad (27)$$

and assume, for simplicity, that the spectrum is a monotonously decreasing function of k , from k_0 ($k_0 \ll k_1$) to infinity. Then, for the interaction of waves (ω_1, k_1) with waves of opposite polarisation, the saturated rate of (26) is the appropriate one, except for wave vectors $k > k_r$ with k_r such that: $P(k_r) = \varepsilon^4 P(k_1)$ where we must apply the unsaturated rate. For the waves (ω_2, k_2) which are polarised like (ω_1, k_1) , we distinguish several regions along the k -axis. We define:

$$k_{c\pm}(k_1, B_1, B_2) = k_1 / \left(1 \pm \frac{\varepsilon}{h} \sqrt{\frac{B_1}{B_2}} \right) \cong k_1 / \left(1 \pm \frac{\varepsilon}{h} \right) \quad (28)$$

where, as before $B_i^2 = \frac{1}{2} P(k_i)$. Then, for $k_{c-} < |k_2| < k_{c+}$, and for $k > k_r$, we must apply the unsaturated rate (26). Since the contributions for $k > k_r$ from different polarisations cancel each other for $\varepsilon \ll 1$, $\Gamma(k_1)$ can be written as a sum involving 5 integrals:

$$\begin{aligned} \Gamma(k_1) = & \frac{1}{2} \sqrt{\frac{\pi}{2}} V_p k_1 g \\ & \cdot \left[- \int_{k_{c-}}^{k_{c+}} \frac{P(k')}{2 B_0^2} (d \log k') + \frac{V_p k_1}{B_0 \Omega_p (P(k_1)/2)^{1/4} h} \right. \\ & \cdot \left\{ - \int_{k_0}^{k_{c+}} \frac{k' (P(k')/2)^{3/4}}{|k_1| - k'} (d \log k') \right. \\ & - \int_{k_{c-}}^{k_0} \frac{k' (P(k')/2)^{3/4}}{k' - |k_1|} (d \log k') \\ & - \int_{k_0}^{|k_1|} \frac{k' (P(k')/2)^{3/4}}{|k_1| + k'} (d \log k') \\ & \left. \left. + \int_{|k_1|}^{k_r} \frac{k' (P(k')/2)^{3/4}}{|k_1| + k'} (d \log k') \right\} \right]. \quad (29) \end{aligned}$$

The first integral in (29) is equal to $P(k_1)^{1/2}/h B_0 \cdot k_1 V_p/\Omega_i$. The others depend somewhat on the form of the spectrum; but it is important to realize that, contrary to the unsaturated case, the main contribution to the damping rate is that of waves of wavenumber whose absolute value is $O(k_1)$. We can

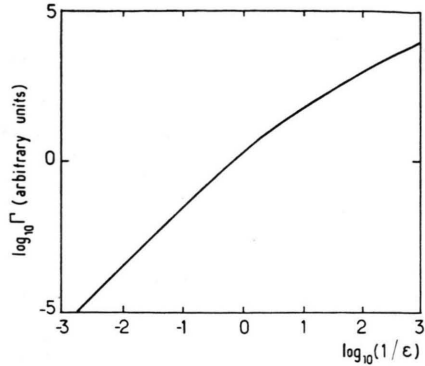


Fig. 1. Schematic diagram of the damping rate of waves (ω_1, k_1) in the presence of a spectrum of waves $P(k) = B_1^2/2 = \text{constant}$, as a function of $(1/\varepsilon) = (\Omega_p/k_1 V_p) (B_1/B_0)$.

write the damping rate as:

$$\Gamma(k_1) = - \frac{1}{2} \sqrt{\frac{\pi}{2}} \frac{g}{h} \frac{(V_p k_1)^2}{\Omega_p} \frac{B_1}{B_0} w, \quad (30)$$

where w depends on h , ε and the shape of the spectrum. Let us consider power law spectra of the form: $P \propto k^{-\alpha}$. For $\alpha = 2/3$; corresponding to a Kolmogorov spectrum, we obtain:

$$w_{2/3} = \frac{1}{h} + \sqrt{2} \log \left(\frac{4h}{\varepsilon} \right). \quad (31)$$

For a Kraichnan [16] spectrum, $\alpha = 1/3$, and

$$w_{1/3} = \frac{1}{h} + \sqrt{2} \log \left(\frac{8h}{\varepsilon} \right) - \frac{1}{\sqrt{2}} \left(\frac{2 + \sqrt{2}}{2 - \sqrt{2}} \right). \quad (32)$$

For a flat spectrum: $P(k) = \text{constant}$ or $\alpha = 0$;

$$w_0 = \frac{1}{h} + \frac{1}{\sqrt{2}} \log \left(\frac{h^2}{\varepsilon^2} \right). \quad (33)$$

To illustrate the effect of the trapping of the particles in wave packets on the nonlinear damping rate Γ of parallel propagating Alfvén waves, we show in Fig. 1 the variations of Γ with $(1/\varepsilon)$, for the case of a spectrum $P(k) = \text{constant}$. The reduction of Γ , at high wave amplitude, is apparent.

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